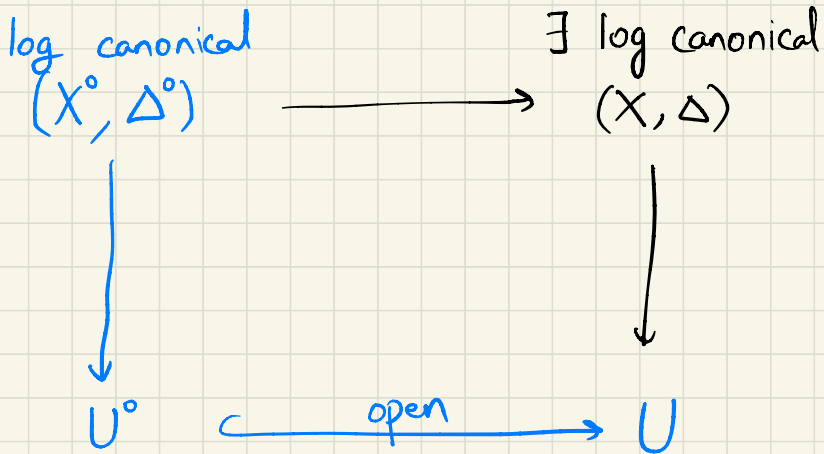


# Main results of the Paper



Won't talk about this or the other main results.  
Will jump straight to preliminaries

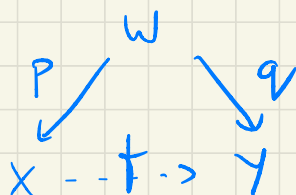
Existence of lc flips.

## Recall some definitions

$\phi: X \dashrightarrow Y$  proper birational contraction

$\begin{matrix} \cup \\ D \end{matrix} \mapsto \begin{matrix} \cup \\ D' \end{matrix}$  divisors

Defn:  $\phi$  is  $D$ -non-positive (resp.  $D$ -negative) if



$W$  = common  
resoln.

- $p^* D = q^* D' + E \quad E \geq 0$
- $p_* E \subseteq \{f\text{-exceptional divisors}\}$   
(resp.  $p_* E = \{f\text{-exceptional divisors}\}$ )

Ex:  $K_X + \Delta$  - non positive = Discr. decreases for  
some  $f$ -exc. divisors

$K_X + \Delta$  - negative = Discr. decreases for  
all  $f$ -exc. divisors

Defn:  $(X, \Delta) \xrightarrow{\phi} (X_M, \Delta_M)$  birational contraction  
 $(X, \Delta)$  and  $(X_M, \Delta_M)$  are both lc

Say  $\phi$  is a minimal model if:

- $X_M$  is  $\mathbb{Q}$ -factorial
- $\phi$  is  $(K_X + \Delta)$ -negative
- $K_{X_M} + \Delta_M$  is nef

### Modifications

- Say  $\phi$  is weak lc model if  
 $(K_X + \Delta)$ -negative replaced with  $(K_X + \Delta)$ -non-positive
- Say  $\phi$  is good minimal model if  
 $K_{X_M} + \Delta_M$  is additionally semiample.

## Lemma 2.2:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & X' \\ h \downarrow & & \downarrow h' \\ Y & \xrightarrow{\eta} & Y' \end{array}$$

Vertical = Fibrations

Horizontal = Bir. contraction

$D$  vertical divisor on  $X$  s.t.  $D \sim h^*E$

[Everything is  $\mathbb{Q}$ -Cartier]

If  $\mu_*D \sim_{Y'} 0$ , then  $\mu_*D \sim h'^*(\eta_*E)$

Pf: First observe: if  $T \sim h^*(B)$  with  $T$  vertical divisor on  $X$ , then  $\exists B'$  s.t.:

$$T = h^*(B')$$

Pf:  $T - h^*(B) = \text{div}(g)$  for  $g$  rat'l fn. on  $X$   
 $g$  doesn't vanish / have poles on gen. fiber

$\therefore g = h^*(g')$  for  $g'$  rat'l fn. on  $Y$

Thus  $T = h^*(B + \text{div}(g'))$



So, replace  $E$  to assume  $D = h^*E$ .

$$\text{Set } F := \underbrace{\mu_* D - h'^*(\eta_* E)}_{\text{differ over points of } Y' \text{ where } \eta^{-1} \text{ is not defined}} \sim_Y 0$$

differ over points of  $Y'$  where  $\eta^{-1}$  is not defined

$\therefore$  Image of  $F$  in  $Y$  is of  $\text{codim} \geq 2$

By prev. page, can get:

$$F = h'^*(E)$$

$$\hookrightarrow \therefore F = 0$$



## Minimal Models

I will frequently skip writing/saying over  $U$

### Thm 2.3 [BCHM's MMP]

$X \dashrightarrow U$  projective morphism

$(X, \Delta)$  dlt

$S = \lfloor \Delta \rfloor$  non Klt locus

If (1)  $\Delta$  big, no strata of  $S$  contained in  $B_+(\Delta/U)$

(2)  $K_X + \Delta$  big, ——— " ———  $B_+(K_X + \Delta/U)$

(3)  $K_X + \Delta$  not pseudoeffective.

Then  $(K_X + \Delta)$ -MMP (with scaling) terminates with

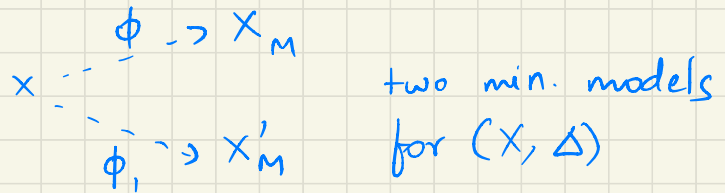
- A good min. model

OR

- Mori Fiber space

# Lemma 2.4 [Comparing two min. models]

$(X, \Delta)$  dlt



$$(1) \{ \phi - \text{Exc. divisors} \} = \{ \text{Divisors in } B_-(K_X + \Delta) \}$$

If  $\phi = \text{good min. model}$

$$\rightarrow = \{ \text{Divisors in } B(K_X + \Delta) \}$$

$$(2) X'_M \dashrightarrow X_M \text{ isom in codim } 1$$

Both have same discrepancies

$$a_E(X'_M, \Delta'_M) = a_E(X_M, \Delta_M) \quad E \text{ divisor on } X$$

$$(3) \phi \text{ good min. model} \Leftrightarrow \phi' \text{ good min. model}$$

Pf: Skip (1) [Use Nakayama-Zariski dec.]

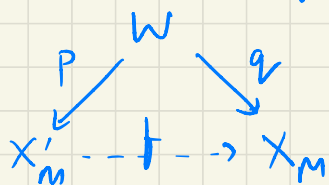
(2)  $\phi$ -exc. divisors =  $\phi'$ -exc. divisors

$\therefore$  Divisors on  $X_M$  = Divisors on  $X'_M$

$\therefore X'_M \dashrightarrow X_M$  iso in codim 1

Claim: Discrep. of  $(X'_M, \Delta'_M)$  and  $(X_M, \Delta_M)$  are same

Pf:



$$p^*(K_{X'_M} + \Delta'_M) + \sum a_E(X'_M, \Delta'_M) E$$

"

$K_W$

"

$$q^*(K_{X_M} + \Delta_M) + \sum a_E(X_M, \Delta_M) E$$

$$D := \underbrace{p^*(K_{X'_M} + \Delta'_M)}_{p\text{-trivial}} - \underbrace{q^*(K_{X_M} + \Delta_M)}_{\text{nef [it is pullback of nef]}} \text{ is } -p\text{-nef}$$

$p_*(D) = 0$ . Negativity lemma  $\Rightarrow D$  effective

Similarly  $-D$  effective.  $\therefore D = 0$

$$\text{i.e. } p^*(K_{X'_M} + \Delta'_M) = q^*(K_{X_M} + \Delta_M)$$

$\Rightarrow$  Discrep are same.

One of them semiamp  $\Leftrightarrow$  Other is semiamp

$\therefore \phi$  good min. model  $\Leftrightarrow \phi'$  good min. model



Lemma 2.5 [Partial converse to previous]

$(X, \Delta)$  dlt

$\phi: X \dashrightarrow X'$  birational contraction s.t.

- $K_{X'} + \Delta'$  is nef
- $\{\phi\text{-Exc divisors}\} = \{\text{Divisors in } B(K_X + \Delta)\}$

Additionally say  $\exists$  good min. model  $\psi: X \dashrightarrow X_M$

Then  $\phi$  is a min. model.

Pf: WTS:  $\phi$  is  $K_X + \Delta$ -negative

$\psi: X \dashrightarrow X_M$  good min. model. Lemma 2.4  $\Rightarrow$

$$\{\psi\text{-Exc divisors}\} = \{\text{Divisors in } B(K_X + \Delta)\}$$

$\therefore X' \dashrightarrow X_M$  is iso in codim 1

$\therefore (X', \Delta')$  and  $(X_M, \Delta_M)$  have same discrep.

$\therefore$  Since  $X \dashrightarrow X_M$  is  $K_X + \Delta$ -negative

$X \dashrightarrow X'$  is also  $K_X + \Delta$ -negative  $\square$

## Lemma 2.6 [Openness of good minimal models]

$(X, \Delta)$  and  $(X', \Delta')$  s.t.

$(X, \Delta_t := (1-t)\Delta + t\Delta')$  is dlt  $0 \leq t < 1$

$K_X + \Delta$  semiample  $\leadsto$  Get morphism

$$g: X \longrightarrow \mathbb{Z}$$

If  $(X, \Delta')$  admits good min. model  $h: X \dashrightarrow X_M$  over  $\mathbb{Z}$ , then  $h$  is a min. model for  $(X, \Delta_t)$  for small  $t$ .

Proof: WTS: ①  $h$  is  $K_X + \Delta_t$ -negative

②  $K_{X_M} + \Delta_{t,M}$  is nef

①  $h$  is  $K_X + \Delta'$ -negative by defn.

$h$  is  $K_X + \Delta$  trivial  $X \dashrightarrow X_M$   
 $\searrow \swarrow$   
 $\mathbb{Z}$

$$K_X + (1-t)\Delta + t\Delta' = \underbrace{(1-t)(K_X + \Delta)}_{\text{trivial}} + \underbrace{t(K_X + \Delta')}_{\text{negative}}$$

$\therefore h$  is  $K_X + \Delta_t$ -negative.

② Let  $K_X + \Delta \sim g^* H$ ,  $H$  ample on  $Z$

Set  $m :=$  Positive integer s.t.  $mH$  Cartier

Assume  $K_{X_m} + \Delta_{t,m}$  not nef

Say curve  $\Sigma$ .  $(-|-) < 0$

As  $K_{X_m} + \Delta_m$  is nef,  $K_{X_m} + \Delta'_m$  must also intersect  $\Sigma$  negatively. Can also assume:

$$0 < -(K_{X_m} + \Delta'_m) \cdot \Sigma < 2 \dim(X)$$

Now:

$$0 > (K_{X_m} + \Delta_{t,m}) \cdot \Sigma$$

$$= t(K_{X_m} + \Delta'_m) \cdot \Sigma + (1-t)(K_{X_m} + \Delta_m) \cdot \Sigma$$

$$\geq -t \cdot 2 \dim(X) + (1-t) \underbrace{H \cdot g_* \Sigma}_{> \frac{1}{m}}$$

[Remember  $g_* \Sigma$  non-zero as  $K_{X_m} + \Delta'_m$  is nef over  $Z$ ]

$$> -t \cdot 2 \dim(X) + (1-t) \frac{1}{m}$$

$$> 0 \text{ (if } t \text{ is sufficiently small)}$$



Lemma 2.7 [Restatement of MMP with scaling of  $A$  terminates]

$(X, \Delta)$  dlt

$A$  = ample divisor on  $X$

TFAE:

①  $R(X; K_X + \Delta, K_X + \Delta + A)$  is fin. generated

Ring generated by global sections of  $K_X + \Delta$  and global sections of  $K_X + \Delta + A$ .

②  $(K_X + \Delta)$  - MMP with scaling of  $A$  terminates and  $(X, \Delta)$  has a good minimal model.

Pf: Skip

Cor 2.9 [MMP with scaling of  $A$  terminates for a dlt pair if  $\exists$  good min. model]

$(X, \Delta)$  dlt,  $Q$ -factorial

If  $\exists$  good min. model of  $(X, \Delta)$ , then:

Any  $(K_X + \Delta)$ -MMP with scaling of  $A$  terminates

Pf: Skip

[Part of the proof same as Proof of Lemma 2.11 below]

Lemma 2.10 [Having good min. models preserved under birational morphism]

$(X, \Delta)$  dlt

$\mu: X' \rightarrow X$  proj. birational morphism

Write  $K_{X'} + \Delta' = \mu^*(K_X + \Delta) + F$

[where  $\Delta', F$  effective with no common comp.]

$(X, \Delta)$  has good min. model  $\Leftrightarrow$

$(X', \Delta')$  has good min. model

Proof: Will show  $(\Rightarrow)$

Suppose  $(X, \Delta)$  has good min. model  $\phi: X \dashrightarrow X_M$

Let  $E \subseteq X'$  defined as:

$$E = \left\{ \mu\text{-exc. div } E \text{ s.t. } a_E(X, \Delta) \leq 0 \text{ and center of } E \text{ not in } B(K_X + \Delta) \right\}$$

These hypotheses on  $E$  implies we can

extract  $E$  i.e. by BCHM 1.4.3,  $\exists X'_M$  s.t.:

$$\begin{array}{ccc} X' & \xrightarrow{\phi'} & X'_M \\ \mu \downarrow & & \downarrow \mu_M \\ X & \xrightarrow{\phi} & X_M \end{array}$$

with Exc. div of  $\mu_M = E$

Observe that no component of  $F$  is in  $E$

as  $a_{F_i}(X, \Delta) > 0$  for every  $F_i \subseteq F$

So,  $\phi'$  has to contract  $F$  [as composition contracts  $F$  but  $\mu_M$  doesn't]

Claim:  $X' \xrightarrow{\phi'} X'_M$  is a good min. model for  $(X', \Delta')$ .

Proof: ①  $K_{X'_M} + \Delta'_M$  is semiample:

$$\begin{aligned} K_{X'_M} + \Delta'_M &= \phi'_*(K_{X'} + \Delta') \\ &= \phi'_*(\mu^*(K_X + \Delta) + F) \\ &= \phi'_*(\mu^*(K_X + \Delta)) + \underbrace{\phi'_*(F)}_{=0} \\ &= \mu'^*_M(\phi_*(K_X + \Delta)) \quad [\text{Lemma 2.2}] \\ &= \mu'^*_M(\underbrace{K_{X_M} + \Delta_M}_{\text{semiample}}) \end{aligned}$$

②  $\phi'$  will be  $K_{X'} + \Delta'$ -negative by choice of  $E$ .



## Lemma 2.11

$(X, \Delta)$  dlt

$X \dashrightarrow \phi \dashrightarrow X_M$  good min. model for  $(X, \Delta)$

If  $\psi: X \dashrightarrow Y := \text{Proj}(R(K_X + \Delta))$  is a morphism then:

There exists a good min. model for  $(X, \Delta)$  over  $Y$ .

Pf: Take common resolution:

$$\begin{array}{ccc} X' & & \\ \mu \downarrow & \searrow \nu & \\ X & \dashrightarrow \phi \dashrightarrow & X_M \end{array}$$

Suffices to show that  $(X', \Delta')$  has good min. model over  $Y$  (by Lemma 2.10)

$$K_{X'} + \Delta' = \mu^*(K_X + \Delta) + F$$

Also as  $\phi$  is  $K_X + \Delta$ -negative

$$\mu^*(K_X + \Delta) = \nu^*(K_{X_M} + \Delta_M) + E$$

$$\therefore K_{X'} + \Delta' = \nu^*(K_{X_M} + \Delta_M) + E + F$$

Claim: ①  $E, F \in B_{\frac{K_{X'} + \Delta'}{X_M}} = \bigcup_{A \text{ ample}} B(K_{X'} + \Delta' + A/X_M)$   
"Diminished base locus"

② Run a  $K_{X'} + \Delta'$ -MMP over  $X_M$  with scaling to get  $\phi': X' \dashrightarrow X'_M$

Then everything in the diminished base locus is contracted.

In particular,  $\phi'_*(E+F) = 0$

$$\therefore K_{X'_M} + \Delta'_M = \phi'_*(K_{X'} + \Delta')$$

Same as in the proof of Lemma 2.10  $\leftarrow$

$$= \underbrace{M_M'^*(K_{X_M} + \Delta_M)}_{\text{semiample}}$$

$\therefore (X'_M, \Delta'_M)$  good min. model of  $(X', \Delta')$



Claim follows from this lemma

Lemma: Let  $p: X \rightarrow Y$  birational morphism

①  $B_-(p^*D + E/Y) \supseteq E$  for  $E$  exceptional

②  $K_X + \Delta_X$ -MMP over  $Y$  with scaling contracts divisors in  $B_-(K_X + \Delta_X/Y)$

Pf: ① We'll prove  $E$  lies in the base locus of  $p^*D + E$

$$\text{Say } p^*D + E \sim_Y F \geq 0$$

$$\Rightarrow E \sim_Y F \geq 0$$

$$\Rightarrow 0 \sim_Y F - E$$

$$\therefore F - E \text{ is } -p\text{-nef and } p_*(F - E) = p_*F \geq 0$$

$$\therefore \text{Negativity lemma} \Rightarrow (F - E) \geq 0$$

$$\text{i.e. } E \leq F$$

② Say  $(K_X + \Delta)$ -MMP over  $Y$  yields  $(X_m, \Delta_m)$

Assume for sake of contradiction that some  $E \in B_-$  is not contracted in  $X_m$ .


Say  $E \in B(K_X + \Delta + \frac{A}{Y})$  for some ample divisor  $A$  over  $Y$

Pushforward to get  $\underbrace{K_{X_m} + \Delta_m}_{\text{nef}} + \underbrace{A_m}_{\text{ample over complement of codim} \geq 2}$

$\therefore K_{X_m} + \Delta_m + A_m$  doesn't contain  $\text{image}(E)$  in its base locus as by assumption  $\text{image}(E)$  has  $\text{codim } 1$ .

$\therefore \exists$  section  $s$  of  $K_{X_m} + \Delta_m + A_m$  which doesn't vanish along  $\text{image}(E)$

Now the pullback of  $s$  will give a section of  $K_X + \Delta + A$  which doesn't vanish along  $E$ . [Remember that by assumption

$X \dashrightarrow X_m$  is isom about  $E$  as  $E$  is not contracted. So, the pullback of  $K_{X_m} + \Delta_m + A_m$  and  $K_X + \Delta + A$  agree about  $E$ ] 



Theorem 2.12 [Finite gen. of canonical ring +  
Very general fiber has min. model  
 $\Rightarrow (X, \Delta)$  dlt has min. model]

$X \xrightarrow{f} U$

$(X, \Delta)$  dlt

If ① For very general  $u \in U$ ,  $(X_u, \Delta_u)$  has a  
good min. model

②  $R(X_U; K_X + \Delta)$  is finitely generated

Then  $(X, \Delta)$  has a good min. model over  $U$

Proof: skip.

Cor 2.13:

$$(Z, \Delta_Z)$$

$\downarrow h = \text{fibration}$  s.t.  $h^*(K_Y + \Theta) \sim K_Z + \Delta_Z$

$$(Y, \Theta)$$

Let  $\eta: Y \dashrightarrow Y'$  be a  $K_Y + \Theta$  flip or a divisorial contraction.

Then  $\exists$  birational contraction:

$$(Z, \Delta_Z) \xrightarrow{\mu} (Z', \Delta_{Z'})$$

$$\downarrow h$$

$$\downarrow h' \text{ s.t. } h'^*(K_{Y'} + \Theta') = K_{Z'} + \Delta_{Z'}$$

$$(Y, \Theta) \xrightarrow{\eta} (Y', \Theta')$$

Pf: Once we prove  $\exists Z'$  and  $\exists$  map  $h'$ , then by Lemma 2.2, we have:

$$h'^*(K_{Y'} + \Theta') = K_{Z'} + \Delta_{Z'}$$

So, we just have to show  $\exists Z' \xrightarrow{h} Y'$

We'll do the case  $\eta = \text{flip}$

$$\begin{array}{ccc}
 Z & \dashrightarrow & Z' \\
 \downarrow & & \downarrow \\
 Y & \dashrightarrow & Y' = \text{Proj}(R(Y/W, K_Y + \Theta)) \\
 & \searrow & \swarrow \\
 & W & 
 \end{array}$$

flipping cont. fin-gen.

Observe ①  $R(Z/W, K_Z + \Delta_Z) = R(Y/W, K_Y + \Theta)$   
 since  $Z \rightarrow Y$  is a fibration  
 So this is fin-gen.

② A general fiber of  $Z \rightarrow W$   
 is also a general fiber of  $Z \rightarrow Y$   
 $\therefore (Z_t, \Delta_{Z_t})$  has a good min model  
 as  $K_{Z_t} + \Delta_{Z_t} \sim 0$

So by Lemma 2.12,  $(Z, \Delta_Z)$  has a good min. model  $Z'$  over  $W$  and it maps to  $\text{Proj}(R(Z'_W, K_{Z'} + \Delta_{Z'}))$

Observe

$$\begin{aligned}
 & \text{Proj}(R(Z'_W, K_{Z'} + \Delta_{Z'})) \\
 &= \text{Proj}(R(Z/W, K_Z + \Delta_Z)) \\
 &= \text{Proj}(R(Y/W, K_Y + \Theta)) \\
 &= Y'
 \end{aligned}$$