Main results of the Paper

$$\begin{tikzcd} \text{log canonical } (X^\circ, \Delta^\circ) \arrow[r] \arrow[d] & \exists \text{ log canonical } (X, \Delta) \arrow[d] \\ U^\circ \arrow[r, hook, "\text{open}"] & U \end{tikzcd}$$

Won't talk about this or the other main results.
Will jump straight to preliminaries

Existence of lc flips.

Recall some definitions

$\phi : X \dashrightarrow Y$ proper birational contraction

$$\begin{array}{ccc} X & \dashrightarrow & Y \\ \cup\!\!\mid & & \cup\!\!\mid \\ D & \longmapsto & D' \end{array}$$

divisors

Defn: ϕ is D-non-positive (resp. D-negative) if

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

W = common resoln.

- $p^* D = q^* D' + E \qquad E \geq 0$
- $p_* E \subseteq \{f\text{-exceptional divisors}\}$
  (resp. $p_* E = \{f\text{-exceptional divisors}\}$)

Ex: $K_X + \Delta$ - non positive = Discrep. decreases for <u>some</u> f-exc. divisors

$K_X + \Delta$ - negative = Discrep. decreases for <u>all</u> f-exc. divisors

<u>Defn</u>: $(X,\Delta) \overset{\phi}{\dashrightarrow} (X_M, \Delta_M)$ birational contraction (X,Δ) and (X_M, Δ_M) are both lc

Say ϕ is a <u>minimal model</u> if:

- X_M is $\mathbb{Q}$-factorial
- ϕ is $(K_X + \Delta)$-negative
- $K_{X_M} + \Delta_M$ is nef

<u>Modifications</u>

- Say ϕ is <u>weak lc model</u> if $(K_X+\Delta)$-negative replaced with $(K_X+\Delta)$-non-positive
- Say ϕ is <u>good minimal model</u> if $K_{X_M} + \Delta_M$ is additionally semiample.

Lemma 2.2:

$$\begin{tikzcd} X \arrow[r, dashed, "\mu"] \arrow[d, "h"'] & X' \arrow[d, "h'"] \\ Y \arrow[r, dashed, "\eta"] & Y' \end{tikzcd}$$

Vertical = Fibrations

Horizontal = Bir. contraction

D vertical divisor on X s.t. $D \sim h^*E$

[Everything is $\mathbb{Q}$-Cartier]

If $\mu_* D \sim_{Y'} 0$, then $\mu_* D \sim h'^*(\eta_* E)$

Pf: First observe: if $\Gamma \sim h^*(B)$ with Γ vertical divisor on X, then $\exists B'$ s.t.:

$$\Gamma = h^*(B')$$

Pf: $\Gamma - h^*(B) = \operatorname{div}(g)$ for g rat'l fn. on X

g doesn't vanish / have poles on gen. fiber

$\therefore g = h^*(g')$ for g' rat'l fn. on Y

Thus $\Gamma = h^*(B + \operatorname{div}(g'))$

So, replace E to assume $D = h^* E$.

Set $F := \underbrace{\mu_* D - h'^*(\eta_* E)}_{\text{differ over points of } Y' \text{ where } \eta^{-1} \text{ is not defined}} \sim_{Y'} 0$

$\therefore$ Image of F in Y is of codim ≥ 2

By prev. page, can get:

$F = h'^*(E)$

$\therefore F = 0$

$\blacksquare$

Minimal Models

I will frequently skip writing/saying over U

Thm 2.3 [BCHM's MMP]

$X \xrightarrow{f} U$ projective morphism

(X, Δ) dlt

$S = \lfloor \Delta \rfloor$ non klt locus

If (1) Δ big, no strata of S contained in $B_+(\Delta/U)$

(2) $K_X + \Delta$ big, ——— " ——— $B_+(K_X + \Delta/U)$

(3) $K_X + \Delta$ not pseudoeffective.

Then $(K_X + \Delta)$-MMP (with scaling) terminates with

- A good min. model

OR

- Mori Fiber space

Lemma 2.4 [Comparing two min. models]

(X, Δ) dlt

$$\begin{tikzcd} & X_M \\ X \arrow[ur, dashed, "\phi"] \arrow[dr, dashed, "\phi_1"'] & \\ & X'_M \end{tikzcd}$$

two min. models for (X, Δ)

(1) $\{\phi\text{-Exc. divisors}\} = \{\text{Divisors in } B_-(K_X+\Delta)\}$

If ϕ = good min. model

$= \{\text{Divisors in } B(K_X+\Delta)\}$

(2) $X'_M \dashrightarrow X_M$ isom in codim 1

Both have same discrepancies

$$a_E(X'_M, \Delta'_M) = a_E(X_M, \Delta_M) \qquad E \text{ divisor on } X$$

(3) ϕ good min. model $\Leftrightarrow$ ϕ' good min. model

Pf : Skip (1) [Use Nakayama-Zariski dec.]

(2) ϕ-exc. divisors = ϕ'-exc. divisors

$\therefore$ Divisors on X_M = Divisors on X_M'

$\therefore$ $X_M' \dashrightarrow X_M$ iso in codim 1

Claim: Discrep. of (X_M', Δ_M') and (X_M, Δ_M) are same

Pf:

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_M' \arrow[rr, dashed, "f"] & & X_M \end{tikzcd}$$

$$p^*(K_{X_M'} + \Delta_M') + \sum a_E(X_M', \Delta_M') E$$
$$\shortparallel$$
$$K_W$$
$$\shortparallel$$
$$q^*(K_{X_M} + \Delta_M) + \sum a_E(X_M, \Delta_M) E$$

$$D := \underbrace{p^*(K_{X_M'} + \Delta_M')}_{p\text{-trivial}} - \underbrace{q^*(K_{X_M} + \Delta_M)}_{\text{nef [it is pullback of nef]}} \text{ is } -p\text{-nef}$$

$p_*(D) = 0$. Negativity lemma $\Rightarrow$ D effective

Similarly $-D$ effective. $\therefore D = 0$

i.e. $p^*(K_{X'_M} + \Delta'_M) = q^*(K_{X_M} + \Delta_M)$

$\Rightarrow$ Discrep are same.

One of them semiamp $\Longleftrightarrow$ Other is semiamp

$\therefore$ ϕ good min. model $\Longleftrightarrow$ ϕ' good min. model

$\blacksquare$

Lemma 2.5 [Partial converse to previous]

(X, Δ) dlt

$\phi : X \dashrightarrow X'$ birational contraction s.t.

- $K_{X'} + \Delta'$ is nef
- $\{\phi\text{-Exc divisors}\} = \{\text{Divisors in } B(K_X + \Delta)\}$

Additionally say $\exists$ good min. model $\psi : X \dashrightarrow X_M$

Then ϕ is a min. model.

Pf: WTS: ϕ is $K_X + \Delta$-negative

$\psi : X \dashrightarrow X_M$ good min. model. Lemma 2.4 $\Rightarrow$

$\{\psi\text{-Exc divisors}\} = \{\text{Divisors in } B(K_X + \Delta)\}$

$\therefore$ $X' \dashrightarrow X_M$ is iso in codim 1

$\therefore$ (X', Δ') and (X_M, Δ_M) have same discrep.

$\therefore$ Since $X \dashrightarrow X_M$ is $K_X + \Delta$-negative

$X \dashrightarrow X'$ is also $K_X + \Delta$-negative $\blacksquare$

Lemma 2.6 [Openness of good minimal models]

(X, Δ) and (X', Δ') s.t.

$$(X, \Delta_t := (1-t)\Delta + t\Delta') \text{ is dlt} \quad 0 \le t < 1$$

$K_X + \Delta$ semiample $\rightsquigarrow$ Get morphism

$$g : X \longrightarrow Z$$

If (X, Δ') admits good min. model $h : X \dashrightarrow X_M$ over Z, then h is a min. model for (X, Δ_t) for small t.

Proof: WTS: ① h is $K_X + \Delta_t$ - negative

② $K_{X_M} + \Delta_{t,M}$ is nef

① h is $K_X + \Delta'$ - negative by defn.

h is $K_X + \Delta$ trivial

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X_M \arrow[dl] \\ & Z & \end{tikzcd}$$

$$K_X + (1-t)\Delta + t\Delta' = \underbrace{(1-t)(K_X + \Delta)}_{\text{trivial}} + \underbrace{t(K_X + \Delta')}_{\text{negative}}$$

$\therefore$ h is $K_X + \Delta_t$ - negative.

(2) Let $K_X + \Delta \sim g^* H$, H ample on Z

Set $m :=$ Positive integer s.t. mH Cartier

Assume $K_{X_M} + \Delta_{t,M}$ not nef

Say curve $\Sigma \cdot (\text{---}''\text{---}) < 0$

As $K_{X_M} + \Delta_M$ is nef, $K_{X_M} + \Delta'_M$ must also intersect Σ negatively. Can also assume:

$$0 < -(K_{X_M} + \Delta'_M) \cdot \Sigma < 2\dim(X)$$

Now:

$$0 > (K_{X_M} + \Delta_{t,M}) \cdot \Sigma$$

$$= t(K_{X_M} + \Delta'_M) \cdot \Sigma + (1-t)(K_{X_M} + \Delta_M) \cdot \Sigma$$

$$\geq -t \cdot 2\dim(X) + (1-t)\underbrace{H \cdot g_* \Sigma}_{> \frac{1}{m}}$$

[Remember $g_* \Sigma$ non-zero as $K_{X_M} + \Delta'_M$ is nef over Z]

$$> -t \cdot 2\dim(X) + (1-t)\frac{1}{m}$$

$$> 0 \quad (\text{if } t \text{ is sufficiently small})$$

Lemma 2.7 [Restatement of MMP with scaling of A terminates]

(X, Δ) dlt

A = ample divisor on X

TFAE:

① $R(X; K_X+\Delta, K_X+\Delta+A)$ is fin. generated

Ring generated by global sections of $K_X+\Delta$ and global sections of $K_X+\Delta+A$.

② $(K_X+\Delta)$-MMP with scaling of A terminates and (X, Δ) has a good minimal model.

Pf: Skip

Cor 2.9 [MMP with scaling of A terminates for a dlt pair if $\exists$ good min. model]

(X,Δ) dlt, $\mathbb{Q}$-factorial

If $\exists$ good min. model of (X,Δ), then:

Any $(K_X+\Delta)$-MMP with scaling of A terminates

Pf: Skip

[Part of the proof same as Proof of Lemma 2.11 below]

Lemma 2.10 [Having good min. models preserved under birational morphism]

(X,Δ) dlt

$\mu : X' \longrightarrow X$ proj. birational morphism

Write $K_{X'} + \Delta' = \mu^*(K_X + \Delta) + F$

[where Δ', F effective with no common comp.]

(X,Δ) has good min. model $\Longleftrightarrow$ (X',Δ') has good min. model

Proof: Will show $(\Rightarrow)$

Suppose (X,Δ) has good min. model $\phi : X \dashrightarrow X_M$

Let $\mathcal{E} \subseteq X'$ defined as:

$$\mathcal{E} = \{\mu\text{-exc. div } E \text{ s.t. } a_E(X,\Delta) \leq 0 \text{ and center of } E \text{ not in } B(K_X + \Delta)\}$$

These hypotheses on $\mathcal{E}$ implies we can extract $\mathcal{E}$ i.e. by BCHM 1.4.3, $\exists X'_M$ s.t.:

$$\begin{tikzcd} X' \arrow[r, dashed, "\phi'"] \arrow[d, "\mu"'] & X'_M \arrow[d, "\mu_M"] \\ X \arrow[r, dashed, "\phi"] & X_M \end{tikzcd}$$

with Exc. div of $\mu_M = \mathcal{E}$

Observe that no component of F is in $\mathcal{E}$ as $a_{F_i}(X, \Delta) > 0$. for every $F_i \subseteq F$

So, ϕ' has to contract F [as composition contracts F but μ_M doesn't]

<u>Claim</u>: $X' \overset{\phi'}{\dashrightarrow} X'_M$ is a good min. model for (X', Δ').

<u>Proof</u>: ① $K_{X'_M} + \Delta'_M$ is semiample:

$$
\begin{aligned}
K_{X'_M} + \Delta'_M &= \phi'_*(K_{X'} + \Delta') \\
&= \phi'_*(\mu^*(K_X + \Delta) + F) \\
&= \phi'_*(\mu^*(K_X + \Delta)) + \underbrace{\phi'_*(F)}_{=0} \\
&= \mu_M'^*(\phi_*(K_X + \Delta)) \quad \text{[Lemma 2.2]} \\
&= \mu_M'^*(\underbrace{K_{X_M} + \Delta_M}_{\text{semiample}})
\end{aligned}
$$

② ϕ' will be $K_{X'} + \Delta'$ - negative by choice of $\mathcal{E}$.

$\square$

Lemma 2.11

(X, Δ) dlt

$X \overset{\phi}{\dashrightarrow} X_M$ good min. model for (X, Δ)

If $\psi : X \dashrightarrow Y := \mathrm{Proj}(R(K_X + \Delta))$ is a morphism then:

There exists a good min. model for (X, Δ) over Y.

Pf: Take common resolution:

$$\begin{tikzcd} X' \arrow[d, "\mu"'] \arrow[dr, "\nu"] & \\ X \arrow[r, dashed, "\phi"] & X_M \end{tikzcd}$$

Suffices to show that (X', Δ') has good min. model over Y (by Lemma 2.10)

$$K_{X'} + \Delta' = \mu^*(K_X + \Delta) + F$$

Also as ϕ is $K_X + \Delta$ - negative

$$\mu^*(K_X + \Delta) = \nu^*(K_{X_M} + \Delta_M) + E$$

$$\therefore \; K_{X'} + \Delta' = \nu^*(K_{X_M} + \Delta_M) + E + F$$

Claim: ① $E, F \subseteq B_-(K_{X'} + \Delta' / X_M) = \bigcup_{A \text{ ample}} B(K_{X'} + \Delta' + A / X_M)$

"Diminished base locus"

② Run a $K_{X'}+\Delta'$-MMP over X_M with scaling to get $\phi' : X' \dashrightarrow X'_M$

Then everything in the diminished base locus is contracted.

In particular, $\phi'_*(E+F)=0$

$$\therefore K_{X'_M} + \Delta'_M = \phi'_*(K_{X'}+\Delta')$$

Same as in the proof of Lemma 2.10

$$= \mu_M'^{*}\underbrace{(K_{X_M}+\Delta_M)}_{\text{semiample}}$$

$\therefore (X'_M, \Delta'_M)$ good min. model of (X', Δ') $\boxed{/\!/}$

Claim follows from this lemma

<u>Lemma</u>: Let $p: X \longrightarrow Y$ birational morphism

① $B_-(p^*D + E/_Y) \supseteq E$ for E exceptional

② $K_X+\Delta_X$ - MMP over Y with scaling contracts divisors in $B_-(K_X+\Delta_X/_Y)$

Pf: ① We'll prove E lies in the base locus of $p^*D + E$

Say $p^*D + E \sim_Y F \geq 0$

$\Rightarrow E \sim_Y F \geq 0$

$\Rightarrow 0 \sim_Y F - E$

$\therefore$ $F - E$ is $-p$-nef and $p_*(F-E) = p_*F \geq 0$

$\therefore$ Negativity lemma $\Rightarrow (F-E) \geq 0$

i.e. $E \subseteq F$

② Say $(K_X + \Delta)$-MMP over Y yields (X_M, Δ_M)

Assume for sake of contradiction that some $E \subseteq B\_$ is not contracted in X_M.

Say $E \subseteq B(K_X + \Delta + A/Y)$ for some ample divisor A over Y

Pushforward to get $\underbrace{K_{X_M} + \Delta_M}_{\text{nef}} + \underbrace{A_M}_{\text{ample over complement of codim} \geq 2}$

$\therefore$ $K_{X_M} + \Delta_M + A_M$ doesn't contain image(E) in its base locus as by assumption image(E) has codim 1.

$\therefore$ $\exists$ section s of $K_{X_M} + \Delta_M + A_M$ which doesn't vanish along image(E)

Now the pullback of s will give a section of $K_X + \Delta + A$ which doesn't vanish along E. [Remember that by assumption $X \dashrightarrow X_M$ is isom about E as E is not contracted. So, the pullback of $K_{X_M} + \Delta_M + A_M$ and $K_X + \Delta + A$ agree about E] $\blacksquare$

Theorem 2.12 [Finite gen. of canonical ring + Very general fiber has min. model $\Rightarrow$ (X, Δ) dlt has min. model]

$X \xrightarrow{f} U$

(X, Δ) dlt

If ① For very general $u \in U$, (X_u, Δ_u) has a good min. model

② $R(X/U; K_X + \Delta)$ is finitely generated

Then (X, Δ) has a good min. model over U

Proof: Skip.

Cor 2.13:

$$\begin{array}{c}(Z, \Delta_Z) \\ \big\downarrow h \\ (Y, \theta)\end{array} \quad h = \text{fibration} \quad \text{s.t.} \quad h^*(K_Y + \theta) \sim K_Z + \Delta_Z$$

Let $\eta : Y \dashrightarrow Y'$ be a $K_Y + \theta$ flip or a divisorial contraction.

Then $\exists$ birational contraction:

$$\begin{tikzcd}
(Z, \Delta_Z) \arrow[r, dashed, "\mu"] \arrow[d, "h"'] & (Z', \Delta_{Z'}) \arrow[d, "h'"] \\
(Y, \theta) \arrow[r, dashed, "\eta"] & (Y', \theta')
\end{tikzcd}$$

s.t. $h'^*(K_{Y'} + \theta') = K_{Z'} + \Delta_{Z'}$

Pf: Once we prove $\exists$ Z' and $\exists$ map h', then by Lemma 2.2, we have:

$$h'^*(K_{Y'} + \theta') = K_{Z'} + \Delta_{Z'}$$

So, we just have to show $\exists$ $Z' \xrightarrow{h} Y'$

We'll do the case $\eta =$ flip

$$
\begin{tikzcd}
Z \arrow[r, dashed] \arrow[d] & Z' \arrow[d] \\
Y \arrow[r, dashed] \arrow[dr, "\text{flipping cont.}"'] & Y' \arrow[d] \\
 & W
\end{tikzcd}
$$

$$Y' = \mathrm{Proj}\,\underbrace{\left(R(Y/W, K_Y+\Theta)\right)}_{\text{fin. gen.}}$$

Observe ① $R(Z/W, K_Z+\Delta_Z) = R(Y/W, K_Y+\Theta)$

since $Z \to Y$ is a fibration

So this is fin. gen.

② A general fiber of $Z \to W$ is also a general fiber of $Z \to Y$

$\therefore (Z_t, \Delta_{Z_t})$ has a good min model as $K_{Z_t}+\Delta_{Z_t} \sim 0$

So by Lemma 2.12, (Z, Δ_Z) has a good min. model Z' over W and it maps to $\mathrm{Proj}(R(Z'/W, K_{Z'}+\Delta_{Z'}))$

Observe $\mathrm{Proj}(R(Z'/W, K_{Z'}+\Delta_{Z'}))$

$= \mathrm{Proj}(R(Z/W, K_Z+\Delta_Z))$

$= \mathrm{Proj}(R(Y/W, K_Y+\Theta))$

$= Y'$